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Stochastic derivation and solution of simplified radiative transfer using the Fokker-Planck equation

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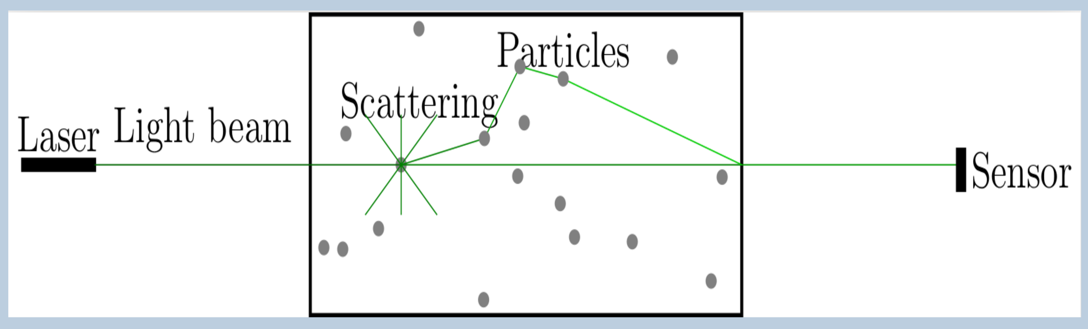
Problem

- Uncertainty Quantification of a stochastic inverse problem
- quantify the uncertainties in optical particle concentration measurements
- we use the Radiative Transfer Equation to model the propagation of light
- our main analytic tool is the Fokker-Planck Equation



Basic concepts

- Radiative Transfer Equation
- Langevin System (stochastic ODE)
- Fokker-Planck Equation (deterministic PDE)
- numerical solution using FDM

Conclusion

The Radiative Transfer Equation was simplified by introducing two new parameters, s_g and s_d, describing the scattering in the measurement direction and in all other directions, respectively. Assuming these two parameters, as well as the absorption coefficient, are stochastic and white noise-based, we derived the corresponding Langevin system. A Fokker-Planck equation was then formulated and solved numerically to precisely quantify the statistics of the light transfer.

References

[1] N. L. Swanson, B. D. Billard, and T. L. Gennaro, *Limits of optical transmission measure- ments with application to particle sizing techniques*. Applied optics, 1999.

[2] H. Risken, *The Fokker-Planck Equation*, volume 18, of Springer Series in Synergetics. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.

[3] C. W. Gardiner, *Stochastic methods, a handbook for the natural and social sciences*. Springer, 2009.

Model

Initially the forward problem is examined for a given particle concentration. The time-independent Radiative Transfer Equation (RTE) [1] describes the rate of light intensity change in the direction z of light propagation:

$$\frac{\partial}{\partial z} I(\vec{r},\hat{z}) = -(a+s)I(\vec{r},\hat{z}) + \frac{s}{4\pi}\int_{\hat{\mu}\in S^2} I(\vec{r},\hat{\mu})\beta(\hat{\mu},\hat{z})d\Omega \tag{1}$$

The intensity change along the z-axis can be approximated by

$$\frac{d}{dz} I = -aI - s_d I + s_g I, \tag{2}$$

which completes a system of Langevin equations:

$$\frac{d}{dz} s_g I = I\Gamma_{s_g}(z) \quad \Rightarrow \quad \frac{d}{dz} s_g = a s_g - s_g^2 + s_g s_d + \Gamma_{s_g}(z) \tag{3}$$

$$\frac{d}{dz} s_d I = I\Gamma_{s_d}(z) \quad \Rightarrow \quad \frac{d}{dz} s_d = a s_d - s_g s_d + s_d^2 + \Gamma_{s_d}(z) \tag{4}$$

$$\frac{d}{dz} a I = I\Gamma_{a}(z) \quad \Rightarrow \quad \frac{d}{dz} a = a^2 - a s_g + a s_d + \Gamma_{a}(z) \tag{5}$$

These equations are used to derive the corresponding Fokker-Planck equation.

The Fokker-Planck equation

From the set of Langevin equations, a Fokker-Planck equation is derived, which has the joint probability density function $W(a, s_g, s_d, I)$ as solution.

$$D^{(1)}(\boldsymbol{x}) = \begin{pmatrix} a^2 - a s_g + a s_d \\ a s_g - s_g^2 + s_g s_d \\ a s_d - s_g s_d + s_d^2 \\ -aI + s_g I - s_d I \end{pmatrix}, \qquad D^{(2)}(\boldsymbol{x}) = \begin{pmatrix} q_a & 0 & 0 & 0 \\ 0 & q_{sg} & 0 & 0 \\ 0 & 0 & q_{sd} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

By using the drift vector $D^{(1)}(\boldsymbol{x})$ and diffusion matrix $D^{(2)}(\boldsymbol{x})$, the Fokker-Planck equation for our Langevin system becomes

$$\begin{aligned} \frac{\partial W}{\partial z}(\boldsymbol{x}, z) &= \left[-\sum_{i=1}^{4} \frac{\partial D_i}{\partial x_i}(\boldsymbol{x}) + \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial^2 D_{ij}}{\partial x_i \partial x_j}(\boldsymbol{x}) \right] W(\boldsymbol{x}, z) \\ &= q_a \frac{\partial^2 W}{\partial a^2}(\boldsymbol{x}, z) + q_{s_g} \frac{\partial^2 W}{\partial s_g^2}(\boldsymbol{x}, z) + q_{s_d} \frac{\partial^2 W}{\partial s_d^2}(\boldsymbol{x}, z) \\ &\quad - \left(a^2 - a s_g + a s_d\right) \frac{\partial W}{\partial a}(\boldsymbol{x}, z) - \left(a s_g - s_g^2 + s_g s_d\right) \frac{\partial W}{\partial s_g}(\boldsymbol{x}, z) \\ &\quad - \left(a s_d - s_g s_d + s_d^2\right) \frac{\partial W}{\partial s_d}(\boldsymbol{x}, z) - \left(-aI + s_g I - s_d I\right) \frac{\partial W}{\partial I}(\boldsymbol{x}, z) \\ &\quad - 3(a + s_g - s_d) W(\boldsymbol{x}, z) \quad \text{in} \quad \Omega. \end{aligned} \tag{7}$$

Here, the domain Ω is a four-dimensional box:

$$\Omega = \{(a, s_g, s_d, I) \mid 0 < a < 1, 0 < s_g < 1, 0 < s_d < 1, 0 < I < I_0\}. \tag{8}$$

Numerical results